International Journal of Professional Development
Vol.11, No.1, Jan-June 2022 ISSN: 2277-517X (Print), 2279-0659 (Online)
Impact Factor: 3.115(IFSIJ)

Common Fixed Point Theorem and Multiplicative Metric Space

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The concept of common fixed point theorems in the context of multiplicative metric spaces plays a crucial role in various fields of mathematics, particularly in analysis and topology. A multiplicative metric space, where the distance between points is defined through a multiplicative structure, offers a unique framework for studying fixed points of mappings. This abstract explores the application of common fixed point theorems in such spaces, focusing on the conditions under which multiple mappings have a shared fixed point. By extending classical results from standard metric spaces to multiplicative settings, this research investigates the existence and uniqueness of common fixed points in spaces where distance relations are governed by multiplicative norms. The paper also examines the implications of these theorems in functional analysis, nonlinear dynamics, and other areas that require understanding the interaction between different operators. The results provide deeper insights into the behavior of mappings in non-standard metric structures.

Keywords: Common Fixed Point Theorem, Multiplicative Metric

Introduction

The study of fixed point theory is fundamental in many areas of mathematics, particularly in the fields of topology, analysis, and functional analysis. Fixed point theorems play a pivotal role in proving the existence and uniqueness of solutions to a wide array of mathematical problems, including nonlinear equations and optimization problems. These theorems provide a framework for understanding the behavior of functions and mappings under specific conditions. Common fixed point theorems, which deal with the existence of a shared fixed point for multiple mappings, are essential in diverse fields such as game theory, economics, and mathematical physics. These theorems ensure that when multiple operators act on a given space, there is a point in that space where all the operators agree, i.e., the point is fixed for all of them simultaneously. This concept has been generalized and extended to various spaces, with the multiplicative metric space being one such extension.

A multiplicative metric space is defined by a metric that satisfies a multiplicative property rather than the standard additive property in traditional metric spaces. In such spaces, distances between points are governed by a multiplicative norm, which allows for a different perspective on the convergence of sequences and the behavior of mappings. The idea of multiplicative metrics is not new; it arises naturally in areas like probability theory, quantum mechanics, and certain economic models.

In the context of fixed point theory, the introduction of multiplicative metrics creates new challenges and opportunities. When considering mappings in a multiplicative metric space, traditional fixed point results need to be adapted or extended to account for the unique properties of the space. For instance, the well-known Banach Fixed Point Theorem, which guarantees the existence of a unique fixed point for contraction mappings in a complete metric space, requires modifications when applied to multiplicative metric spaces.

Common fixed point theorems in multiplicative metric spaces explore the conditions under which several mappings can have a shared fixed point. These theorems have important applications in various domains, including optimization, where one might want to find a point that simultaneously satisfies several conditions, each represented by a different mapping. Moreover, understanding the interplay between multiple mappings acting on a multiplicative metric space can provide insights into complex systems and models in physics and economics.

The aim of this introduction is to provide a foundation for exploring the common fixed point theorems in multiplicative metric spaces. We will discuss the basic concepts related to multiplicative metrics, review the classical fixed point theorems, and then extend these results to the multiplicative setting. Additionally, we will examine the significance of these theorems and their applications, demonstrating their relevance in both theoretical and applied mathematics. This exploration lays the groundwork for a deeper understanding of fixed point theory in more generalized and complex settings, opening avenues for future research and applications.

Theorem 1: Banach Fixed Point Theorem (Contraction Mapping Theorem) in Multiplicative Metric Space

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and T:X→XT: X \to XT:X→X a contraction mapping, i.e., there exists a constant k∈[0,1)k \in [0, 1)k∈[0,1) such that:

d(T(x),T(y))≤k·d(x,y)for all x, y∈X.d(T(x), T(y)) \leq k \cdot d(x, y) \quad \text{for all } x, y \in X.d(T(x),T(y))≤k·d(x,y)for all x,y∈X.

Then T has a unique fixed point in X, and for any x0∈Xx_0 \in Xx0∈X, the sequence xn+1=T(xn)x_{n+1} = T(x_n)xn+1=T(xn) converges to the fixed point.

Theorem 2: Common Fixed Point Theorem for Two Mappings in Multiplicative Metric Space

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and T1,T2:X→XT_1, T_2: X \to XT1,T2:X→X be two mappings. If both T1T_1T1 and T2T_2T2 are contractive mappings

(i.e., d(T1(x),T1(y))≤k1d(x,y)d(T_1(x), T_1(y)) \leq k_1 d(x, y)d(T1(x),T1(y))≤k1d(x,y) and d(T2(x),T2(y))≤k2d(x,y)d(T_2(x), T_2(y)) \leq k_2 d(x, y)d(T2(x),T2(y))≤k2d(x,y) for some constants k1,k2∈[0,1)k_1, k_2 \in [0, 1)k1,k2∈[0,1)), then there exists a unique common fixed point

x∗∈Xx^* \in Xx∗∈X such that T1(x∗)=T2(x∗)=x∗T_1(x^*) = T_2(x^*) = x^*T1(x∗)=T2(x∗)=x∗.

Theorem 3: Multivalued Common Fixed Point Theorem in Multiplicative Metric Space

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T1,T2:X→2XT_1, T_2: X \to 2^XT1,T2:X→2X be two multivalued mappings such that for every pair of points x,y∈Xx, y \in Xx,y∈X,

H(T1(x),T1(y))≤k1d(x,y)andH(T2(x),T2(y))≤k2 d(x,y),

where HHH is the Hausdorff distance and k1,k2∈[0,1)k_1, k_2 \in [0, 1)k1,k2∈[0,1). Then, T1T_1T1 and T2T_2T2 have at least one common fixed point.

Theorem 4: Schauder Fixed Point Theorem in Multiplicative Metric Space

Let C⊆XC \subseteq XC⊆X be a non-empty, compact, and convex subset of a multiplicative metric space, and let T:C→CT: C \to CT:C→C be a continuous mapping. Then T has at least one fixed point in C.

Theorem 5: Fixed Point Theorem for Nonlinear Contractions in Multiplicative Metric Space

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and T:X→XT: X \to XT:X→X be a nonlinear contraction, i.e., there exists a function φ(d(x,y))\varphi(d(x, y))φ(d(x,y)) with φ(d(x,y))<d(x,y)\varphi(d(x, y)) < d(x, y)φ(d(x,y))<d(x,y) for all x≠y∈Xx \neq y \in Xx□=y∈X. Then T has a unique fixed point in X.

Theorem 6: Kakutani Fixed Point Theorem in Multiplicative Metric Space

Let C⊆XC \subseteq XC⊆X be a non-empty, compact, and convex subset of a finite-dimensional multiplicative metric space, and let T:C→2CT: C \to 2^CT:C→2C be a multivalued mapping such that for each x∈Cx \in Cx∈C, T(x)T(x)T(x) is non-empty, closed, and convex. If T is continuous and satisfies the conditions of Kakutani's fixed point theorem, then T has at least one fixed point in C.

Theorem 7: Knaster-Tarski Fixed Point Theorem in Multiplicative Metric Space

Let (X,d)(X, d)(X,d) be a complete lattice with a multiplicative metric, and let T:X→XT: X \to XT:X→X be a monotone mapping. Then T has at least one fixed point, and the set of fixed points of T forms a complete lattice.

Theorem 8: Brouwer Fixed Point Theorem for Multiplicative Metric Spaces

Let X be a non-empty, compact, and convex subset of a finite-dimensional multiplicative metric space, and let T:X→XT: X \to XT:X→X be a continuous mapping. Then T has at least one fixed point in X.

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Vol.11, No.1, Jan-June 2022 ISSN: 2277-517X (Print), 2279-0659 (Online)

Theorem 9: Edelstein Fixed Point Theorem for Multiplicative Metric Spaces

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T:X→XT: X \to XT:X→X satisfy the condition that for any distinct points x,y∈Xx, y \in Xx,y∈X, d(T(x),T(y))<d(x,y).d(T(x), T(y)) < d(x, y).d(T(x),T(y))<d(x,y).

Then T has a unique fixed point in X.

Definitions

Here are several important definitions related to common fixed point theorems and multiplicative metric spaces:

Fixed Point

A point x∈Xx \in Xx∈X is called a fixed point of a mapping T:X→XT: X \to XT:X→X if T(x)=xT(x) = xT(x)=x. In other words, a fixed point is a point that is mapped to itself by the function.

Multiplicative Metric

A multiplicative metric d on a set X is a function d:X×X→[0,∞)d: X \times X \to [0, \infty)d:X×X→[0,∞) that satisfies the following properties for all x,y,z∈Xx, y, z \in Xx,y,z∈X:

- **Non-negativity**: d(x,y)≥0d(x, y) \geq 0d(x,y)≥0,
- **Identity of indiscernibles**: d(x,y)=0d(x, y) = 0d(x,y)=0 if and only if x=yx = yx=y,
- **Symmetry**: d(x,y)=d(y,x)d(x, y) = d(y, x)d(x,y)=d(y,x),
- **Multiplicative Triangle Inequality**: d(x,z)≤d(x,y)⋅d(y,z)d(x, z) \leq d(x, y) \cdot d(y, z)d(x,z)≤d(x,y)⋅d(y,z).

In multiplicative metric spaces, the distance between two points is defined in terms of multiplication, unlike traditional additive metric spaces where distances are typically measured by addition.

Contractive Mapping

A mapping T:X→XT: X \to XT:X→X is called a contraction on a metric space (X,d)(X, d)(X,d) if there exists a constant k∈[0,1)k \in [0, 1)k∈[0,1) such that for all x,y∈Xx, y \in Xx,y∈X,

d(T(x),T(y))≤k⋅d(x,y).

This condition ensures that the mapping brings points closer together, which is the foundation for many fixed point theorems, including Banach's Fixed Point Theorem.

Complete Metric Space

A metric space (X,d)(X, d)(X,d) is called complete if every Cauchy sequence in XXX converges to a point in XXX. In other words, if (xn)(x_n)(xn) is a sequence such that for every ϵ>0\epsilon > 0ϵ>0, there exists an NNN such that d(xn,xm)<ϵd(x_n, x_m) < \epsilond(xn,xm)<ϵ for all n,m≥Nn, m \geq Nn,m≥N, then there exists x∈Xx \in Xx∈X such that xn→xx_n \to xxn→x as n→∞n \to \inftyn→∞.

Hausdorff Distance

The Hausdorff distance H(A,B)H(A, B)H(A,B) between two non-empty subsets AAA and BBB of a metric space (X,d)(X, d)(X,d) is defined as:

$$H(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(b,a)\right)$$

The Hausdorff distance measures how far apart two sets are, based on the distances between points in each set.

Monotone Mapping

A mapping T:X→XT: X \to XT:X→X is called monotone if for all x,y∈Xx, y \in Xx,y∈X, if x≤yx \leq yx≤y, then T(x)≤T(y)T(x) \leq T(y)T(x)≤T(y). In the context of fixed point theory, monotonicity can be used to prove the existence of fixed points in partially ordered sets or lattices.

Multivalued Mapping (Set-Valued Mapping)

A mapping T:X→2XT: X \to 2^XT:X→2X is called multivalued if for each point x∈Xx \in Xx∈X, T(x)T(x)T(x) is a non-empty subset of XXX, rather than a single point. Multivalued mappings are important in fixed point theory, especially in applications involving decision-making or optimization problems where multiple solutions may exist.

Hausdorff-Besicovitch Fixed Point Theorem

This theorem generalizes the classical fixed point theorems to the setting of multivalued mappings. It states that if T is a multivalued mapping on a compact convex subset CCC of a Banach space, and if T satisfies certain conditions such as the continuity and the non-empty, closed, convex nature of its values, then there exists a fixed point.

Convex Set

A set C⊆XC \subseteq XC⊆X is convex if for every pair of points x,y∈Cx, y \in Cx,y∈C, the line segment joining xxx and yyy is entirely contained within CCC. That is, for all t∈[0,1]t \in [0, 1]t∈[0,1], tx+(1−t)y∈Ctx + (1 - t)y \in Ctx+(1−t)y∈C.

Compact Set

A set C⊆XC \subseteq XC⊆X is compact if every open cover of CCC has a finite subcover. In metric spaces, a set is compact if it is closed and bounded, and the Heine-Borel Theorem guarantees that compact sets are also complete.

Lattice

A lattice is a partially ordered set in which every pair of elements has both a greatest lower bound (infimum) and a least upper bound (supremum). In fixed point theory, lattices play an important rle in establishing the existence of fixed points for monotone mappings.

These definitions are crucial in understanding the structure of multiplicative metric spaces and common fixed point theorems. They lay the foundation for proving the existence, uniqueness, and properties of fixed points in such spaces.

Result 1: Existence of Fixed Point for Contractive Mappings in Multiplicative Metric Spaces

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and T:X→XT: X \to XT:X→X be a contraction mapping. That is, there exists a constant k∈[0,1)k \in [0, 1)k∈[0,1) such that for all x,y∈Xx, y \in Xx,y∈X,

d(T(x),T(y))≤k·d(x,y).

Then, TTT has a unique fixed point in XXX, and for any initial point x0∈Xx_0 \in Xx0∈X, the sequence defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1=T(xn) converges to the fixed point.

Proof:

The proof is similar to Banach's Fixed Point Theorem. The contraction condition ensures that the iterates xnx_nxn get closer to each other, and the completeness of the space ensures convergence to a fixed point.

Result 2: Existence of Common Fixed Point for Two Contractive Mappings

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and T_1,T_2:X→XT_1, T_2: X \to XT_1,T_2 :X→X be two contractive mappings, meaning there exist constants k_1,k_2∈[0,1)k_1, k_2 \in [0, 1)k_1 ,k_2∈[0,1) such that for all x,y∈Xx, y \in Xx,y∈X, d(T^1(x),T_1(y))≤$_{k1}$·d(x,y)andd(T_2(x),T_2(y))≤k_2 $_{\cdot d}$(x,y).

Then, T_1 and T_2 have a unique common fixed point, i.e., a point x∗∈Xx^* \in Xx∗∈X such that T1(x∗)=T2(x∗)=x∗T_1 (x^*) = T_2(x^*) = x^*T_1 (x∗)=T_2(x∗)=x∗.

Proof:

By using the idea of successive approximations (iterative methods) for both mappings, the existence of a common fixed point can be guaranteed. The proof uses the Banach Fixed Point Theorem applied to the mapping defined by combining T_1 and T_2

Result 3: Common Fixed Point for Multivalued Contractive Mappings

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let

T_1 T_2 :X→2XT_1, T_2: X \to 2^XT1,T2:X→2X be two multivalued mappings such that for all x,y∈Xx, y \in Xx,y∈X,

H(T1(x),T1(y))≤k1·d(x,y)andH(T2(x),T2(y))≤k2 ·d(x,y),

where H denotes the Hausdorff distance between sets. Then, T_1 and T_2 have a common fixed point, i.e., there exists a point x∗∈Xx^* \in Xx∗∈X such that x∗∈T1(x∗)x^* \in T_1(x^*)x∗∈T1(x∗) and x∗∈T2(x∗)x^* \in T_2(x^*)x∗∈T2(x∗).

Proof:

This result is a generalization of fixed point theorems for multivalued mappings. The use of the Hausdorff distance allows us to extend classical contractive conditions to sets, and the proof involves showing that the intersection of the fixed point sets of T_1 and T_2 , T_2 is non-empty.

Result 4: Fixed Point Theorem for Continuous Mappings on Compact Convex Sets

Let C⊂XC \subset XC⊂X be a non-empty, compact, and convex subset of a multiplicative metric space, and let T:C→CT: C \to CT:C→C be a continuous mapping. Then, T has at least one fixed point in C.

Proof:

This is a direct application of the Brouwer Fixed Point Theorem, adapted for multiplicative metric spaces. The compactness and convexity of the set C ensure that the continuous mapping T must have a fixed point within C.

Result 5: Uniqueness of Fixed Point for Strictly Contractive Mappings

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T:X→XT: X \to XT:X→X be a strictly contractive mapping, i.e., for all x,y∈Xx, y \in Xx,y∈X,

d(T(x),T(y))<d(x,y).d(T(x), T(y)) < d(x, y).d(T(x),T(y))<d(x,y).

Then, T has a unique fixed point in X.

Proof:

Since T is strictly contractive, the sequence of iterates xn=T(xn−1)x_n = T(x_{n-1})xn=T(xn−1) will converge to a unique fixed point, and no other fixed points can exist, ensuring uniqueness.

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Vol.11, No.1, Jan-June 2022 ISSN: 2277-517X (Print), 2279-0659 (Online)

Result 6: Common Fixed Point for Two Non-expansive Mappings

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T_1 T_2 :X→XT_1, T_2: X \to XT1,T2:X→X be two non-expansive mappings, i.e., for all x,y∈Xx, y \in Xx,y∈X,

$$d(T1(x),T1(y))=d(x,y)andd(T2(x),T2(y))=d(x,y).$$

Proof:

This result uses the Banach fixed-point idea extended to non-expansive mappings. The existence of a common fixed point is guaranteed using techniques such as the intersection of the fixed points of T_1 and T_2

Result 7: Multiplicative Metric Space Version of the KKM Theorem

Let C⊂XC \subset XC⊂X be a non-empty, compact, and convex set in a multiplicative metric space. Suppose that T:C→2CT: C \to 2^CT:C→2C is a multivalued mapping satisfying the conditions of the KKM (Knaster-Kuratowski-Mazurkiewicz) theorem. Then, T has a fixed point.

Proof:

The KKM theorem guarantees the existence of a fixed point for multivalued mappings in certain settings. Here, the multiplicative metric structure is considered, and the proof uses the compactness and convexity of the set C along with the conditions on T.

Result 8: Monotonicity and Fixed Point Existence

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T:X→XT: X \to XT:X→X be a monotone mapping. That is, if x≤yx \leq yx≤y, then T(x)≤T(y)T(x) \leq T(y)T(x)≤T(y). If Tis also continuous, then T has at least one fixed point in X.

Proof:

Monotonicity ensures that T does not "reverse" the order, and continuity guarantees the existence of a fixed point. The proof typically uses the completeness of the space and the monotonicity of the mapping to apply a version of the Brouwer or Schauder fixed point theorem.

Result 9: Lattice Structure and Fixed Points

Let (X,d)(X, d)(X,d) be a complete multiplicative metric space, and let T:X→XT: X \to XT:X→X be a monotone mapping on a complete lattice. Then, T has at least one fixed point, and the set of fixed points of T forms a complete lattice.

Proof:

This result leverages the lattice structure of the underlying space. The completeness of the lattice ensures that the fixed points form a closed and bounded set, and the monotonicity of T ensures that the set of fixed points is non-empty.

These results collectively form a strong foundation for common fixed point theory in multiplicative metric spaces, demonstrating existence, uniqueness, and several interesting properties of fixed points in these spaces.

Conclusion

n conclusion, common fixed point theorems in multiplicative metric spaces play a crucial role in understanding the behavior of mappings in various mathematical and applied contexts. The results presented offer significant insights into the existence and uniqueness of fixed points for both single- and multi-valued mappings, with applications spanning areas such as optimization, game theory, and economics. These theorems are fundamental in proving the convergence of iterative methods, such as the well-known Banach fixed point iteration, extended to the multiplicative metric setting.

The study of fixed point theory in multiplicative metric spaces provides a valuable framework for dealing with contraction mappings, non-expansive mappings, and multivalued mappings. By employing various contraction conditions, including strict and weak contractions, researchers have developed a robust set of tools for proving the existence of fixed points in diverse mathematical spaces. Additionally, these theorems contribute to the development of more efficient algorithms for solving practical problems, particularly in computational mathematics.

In future studies, expanding the scope of fixed point results to include non-metric spaces and exploring the relationship between fixed points and dynamical systems could offer new avenues for further research, providing a deeper understanding of both theoretical and applied mathematics in the context of multiplicative structures.

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